

Vizing's 2-factor Conjecture Involving Large Maximum Degree

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Abstract. Let G be a connected simple graph of order n and let $\Delta(G)$ and $\chi'(G)$ denote the maximum degree and chromatic index of G , respectively. Vizing proved that $\chi'(G) = \Delta(G)$ or $\Delta(G) + 1$. Following this result, G is called Δ -critical if $\chi'(G) = \Delta(G) + 1$ and $\chi'(G - e) = \Delta(G)$ for every $e \in E(G)$. In 1968, Vizing conjectured that if G is an n -vertex Δ -critical graph, then the independence number $\alpha(G) \leq n/2$. Furthermore, he conjectured that, in fact, G has a 2-factor. Luo and Zhao showed that if G is an n -vertex Δ -critical graph with $\Delta(G) \geq n/2$, then $\alpha(G) \leq n/2$. More recently, they showed that if G is an n -vertex Δ -critical graph with $\Delta(G) \geq 6n/7$, then G has a hamiltonian cycle, and so G has a 2-factor. In this paper, we show that if G is an n -vertex Δ -critical graph with $\Delta(G) \geq n/2$, then G has a 2-factor.

Keywords. Vizing's 2-factor Conjecture; Edge chromatic index; Tutte's 2-factor Theorem

1 Introduction

In this paper, we only consider simple and finite graphs. Let G be a graph. We fix the notation Δ for the maximum degree of G throughout the paper. A k -vertex of G is a vertex of degree k in G . Denote by V_Δ the set of Δ -vertices in G and by $\chi'(G)$ the edge-chromatic index of G . The graph G is called *critical* (*edge-chromatic critical*) if it has no isolated vertices and $\chi'(G - e) < \chi'(G)$ for every $e \in E(G)$. From the definition, it is clear that if G is critical, then G is connected. In 1965, Vizing [11] showed that a graph of maximum degree Δ has edge chromatic index either Δ or $\Delta + 1$. If $\chi'(G) = \Delta$, then G is said to be of class 1; otherwise, it is said to be of class 2. Appearing easily, however, Holyer [5] showed that it is NP-complete to determine whether an arbitrary graph is of class 1. A critical graph G is called Δ -critical if $\chi'(G) = \Delta + 1$. So Δ -critical graphs are class 2 graphs. On the other hand, every critical class 2 graph of maximum degree Δ is a Δ -critical graph. Motivated by the classification problem, Vizing studied critical class 2 graphs and made the following two well-known conjectures.

The first one, appeared in [12], is on the independence number $\alpha(G)$ of G , that is, the size of a maximum independent set of G .

Conjecture 1 (Vizing's Independence Number Conjecture). *Let G be a Δ -critical graph of order n . Then $\alpha(G) \leq n/2$.*

The second one, appeared in [10], is on 2-factor, a 2-regular spanning subgraph.

Conjecture 2 (Vizing's 2-factor Conjecture). *Let G be a Δ -critical graph. Then G contains a 2-factor.*

As each cycle C satisfying $\alpha(C) \leq |V(C)|/2$, Conjecture 2 implies Conjecture 1. For the Independence Number Conjecture, Brinkmann et al. [2] in 2000 proved that if G is an n -vertex Δ -critical graph, then $\alpha(G) < 2n/3$; and the upper bound is further improved when the maximum degree is between 3 and 10. In 2006, Luo and Zhao [6] confirmed the conjecture for graphs with large maximum degree.

Theorem 1.1. *Let G be an n -vertex Δ -critical graph. Then $\alpha(G) \leq n/2$ if $\Delta \geq n/2$.*

Additionally, Luo and Zhao [7] in 2008 showed that if G is an n -vertex Δ -critical graph, then $\alpha(G) < (5\Delta - 6)n/(8\Delta - 6) < 5n/8$ when $\Delta \geq 6$. In 2009, Woodall [13] further improved the upper bound to $3n/5$. Compared to the progress on Vizing's Independence Number Conjecture, the progress on the 2-factor Conjecture is slower. In 2004, Grünewald and Steffen [4] established Vizing's 2-factor conjecture for graphs with the deficiency $\sum_{v \in V(G)} (\Delta(G) - d_G(v))$ small; in particular, for overfull graphs, i.e., graphs of odd order and with the deficiency $\sum_{v \in V(G)} (\Delta(G) - d_G(v)) < \Delta(G)$. In 2012, Luo and Zhao [8] proved that if G is an n -vertex Δ -critical graph with $\Delta \geq 6n/7$, then G contains a Hamiltonian cycle, and thus a 2-factor with exactly one component. Still considering Δ -critical graphs with large maximum degree, in line with Luo and Zhao's result on the Independence Number Conjecture (Theorem 1.1), in this paper, we reduce the lower bound from $6n/7$ to $n/2$ as follows.

Theorem 1.2. *Let G be an n -vertex Δ -critical graph. Then G has a 2-factor if $\Delta \geq n/2$.*

2 Notations and Lemmas

For a vertex x of a graph G , we denote by $N_G(x)$ the set of neighbors of x in G and by $d_G(x)$ the degree of x in G . For a set of vertices S in G , we define $N_G(S)$ by $N_G(S) = \bigcup_{x \in S} N_G(x)$. For disjoint sets of vertices S and T in G , we denote by $e_G(S, T) = |E_G(S, T)|$, the number of edges that has one end vertex in S and the other in T . If S is a singleton set $S = \{s\}$, we write $e_G(s, T)$ instead of $e_G(\{s\}, T)$. If G is a bipartite graph with partite sets A and B , we denote G by $G[A, B]$ to emphasize the two partite sets. To prove Theorem 1.2, we present a few lemmas.

Lemma 2.1 (Vizing's Adjacency Lemma). *Let G be a Δ -critical graph. Then for any edge $xy \in E(G)$, x is adjacent to at least $\Delta - d_G(y) + 1$ Δ -vertices z with $z \neq y$.*

As there are two specified bipartite graphs $H^*[X', T]$ and $H[X, T]$ defined in the sequel, for consistency, we use notation $H^*[X', T]$ in lemmas only regarding to the bipartite graph $H^*[X', T]$. Denote by $R[A, B]$ for a general bipartite graph in distinguishing with the bipartite graphs H^* and H . A matching of a graph G is a set of independent edges in G . If M is a matching of G , then let $V(M)$ denote the set of end vertices of the edges in M . For $X \subseteq V(G)$, M is said to saturate X if $X \subseteq V(M)$. The following result, which guarantees a matching saturating one partite set in a bipartite graph, can also be found in [3].

Lemma 2.2. *Let H^* be a bipartite graph with partite sets X' and T . If there is no isolated vertex in T and $d_{H^*}(y) \geq d_{H^*}(x)$ for every edge xy with $x \in X'$ and $y \in T$, then H^* has a matching which saturates T .*

Proof. Suppose not. Then by Hall's Theorem, there is a nonempty set $A \subseteq T$ such that $|N_{H^*}(A)| < |A|$. We choose A such that it has the minimum cardinality under the constraint that $|N_{H^*}(A)| < |A|$. Let $B := N_{H^*}(A)$ and $H' := H^*[A \cup B]$ be the subgraph induced by $A \cup B$. We claim that, in H' , there is a matching saturating B . Suppose not. Then by Hall's Theorem again, there is a nonempty subset $B' \subseteq B$ such that $|N_{H'}(B')| < |B'|$. Since $B' \subseteq B = N_{H^*}(A) \neq \emptyset$ (T has no isolated vertices), $N_{H'}(B') \neq \emptyset$. Let $A' = A - N_{H'}(B')$. As $|A| > |B| \geq |B'| > |N_{H'}(B')| > 0$, we have $0 < |A'| < |A|$. On the other hand, we have $N_{H'}(A') = N_{H^*}(A') = B - B'$. So, the sequence of inequalities $|A'| = |A| - |N_{H'}(B')| > |B| - |N_{H'}(B')| > |B| - |B'| = |B - B'| = |N_{H^*}(A')|$ holds, showing a contradiction to the minimality of A under the condition $|N_{H^*}(A)| < |A|$. Let M be a matching which saturates B in H' . Since $|A| > |B|$, $A - V(M) \neq \emptyset$. Let $y^* \in A - V(M)$. Then as T has no isolated vertices, $d_{H^*}(y^*) = d_{H'}(y^*) \geq 1$. Thus,

$$\begin{aligned} e_{H^*}(A, B) &= \sum_{xy \in M, x \in B, y \in A} d_{H'}(x) \quad (M \text{ saturates } B \text{ in } H') \\ &\leq \sum_{xy \in M, x \in B, y \in A} d_{H^*}(x) \\ &\leq \sum_{xy \in M, x \in B, y \in A} d_{H^*}(y) \\ &< \sum_{xy \in M, x \in B, y \in A} d_{H^*}(y) + d_{H^*}(y^*) \\ &\leq e_{H^*}(A, B), \end{aligned}$$

showing a contradiction. ■

Lemma 2.3. *Let R be a bipartite graph with partite sets A and B , $A_1 = \{x \in A \mid d_R(x) = 1\}$ and $B_1 = N_R(A_1)$. Then R has a matching saturating B if the bipartite graph $R'[A - A_1, B - B_1] := R[(A - A_1) \cup (B - B_1)]$ has a matching saturating $B - B_1$.*

Proof. Suppose that R' has a matching M' which saturates $B - B_1$. Since each vertex in A_1 has a unique neighbor in B_1 , there is a matching M_0 saturating B_1 in the subgraph of R induced on $A_1 \cup B_1$. Then $M' \cup M_0$ is a matching which saturates B in R . ■

The following lemma is a generalization of a result in [6].

Lemma 2.4. *Let G be a Δ -critical graph and T be an independent set of G . Let $X' = V(G) - T$ and $H^* := G - E(G[X'])$ be the bipartite graph with partite sets X' and T . Then for each edge $xy \in E(H^*)$ with $x \in X'$ and $y \in T$, $d_{H^*}(y) \geq d_{H^*}(x) + 1 - \delta_0 + \sigma_x$, where $\delta_0 = |T \cap V_\Delta|$ is the number of Δ -degree vertices in T and σ_x is the number of non Δ -degree neighbors of x in X' . Moreover, if $\delta_0 \leq 1$, then there is a matching which saturates T in H^* .*

Proof. Let $xy \in E(H^*)$ with $x \in X'$ and $y \in T$. By Vizing's Adjacency Lemma (Lemma 2.1), x is adjacent to at least $\Delta - d_G(y) + 1$ Δ -vertices in G . As T has δ_0 Δ -vertices, we know x is adjacent to at least $\Delta - d_G(y) + 1 - \delta_0$ Δ -vertices in X' (notice that the quantity is meaningful only if $\Delta - d_G(y) + 1 - \delta_0 > 0$). Then, $\Delta \geq d_G(x) = d_{H^*}(x) + e_G(x, X') \geq d_{H^*}(x) + \Delta - d_G(y) + 1 - \delta_0 + \sigma_x$. Thus $d_{H^*}(y) = d_G(y) \geq d_{H^*}(x) + 1 - \delta_0 + \sigma_x$.

When $\delta_0 \leq 1$, for every edge $xy \in E(H^*)$ with $x \in X'$ and $y \in T$, the inequalities $d_{H^*}(y) \geq d_{H^*}(x) + \sigma_x \geq d_{H^*}(x)$ hold. As G is Δ -critical, it is connected. Consequently, $d_{H^*}(y) = d_G(y) \geq 1$ for each $y \in T$. By applying Lemma 2.2, we see that there is a matching which saturates T in H^* . ■

3 A Detour to Tutte's 2-Factor Theorem

Tutte in [9] obtained a necessary and sufficient condition for a graph to contain an f -factor; the characterization involves pairs of two disjoint vertex sets. Let G be a graph and (S, T) be an ordered pair of disjoint vertex sets of G . A component C of $G - (S \cup T)$ is said to be an *odd component* w.r.t. (S, T) (resp. *even component* w.r.t. (S, T)) if $e_G(C, T) \equiv 1 \pmod{2}$ (resp. $e_G(C, T) \equiv 0 \pmod{2}$). Let $\mathcal{H}_G(S, T)$ be the set of odd components of $G - (S \cup T)$, $h_G(S, T) = |\mathcal{H}_G(S, T)|$, and let $\delta_G(S, T) = 2|S| + \sum_{v \in T} d_{G-S}(v) - 2|T| - h_G(S, T)$. It is easy to see $\delta_G(S, T) \equiv 0 \pmod{2}$ for every $S, T \subseteq V(G)$ with $S \cap T = \emptyset$. We use the following criterion for the existence of a 2-factor, which is a restricted form of Tutte's f -Factor Theorem.

Theorem 3.1. *A graph G has a 2-factor if and only if $\delta_G(S, T) \geq 0$ for every $S, T \subseteq V(G)$ with $S \cap T = \emptyset$.*

An ordered pair (S, T) consists of disjoint sets of vertices S and T in a graph G is called a barrier if $\delta_G(S, T) \leq -2$. By Theorem 3.1, every graph G without a 2-factor has a barrier. A barrier (S, T) is called a minimum barrier if $|S \cup T|$ is smallest among all the barriers of G . A minimum barrier of a graph without a 2-factor has some nice properties, see [1, 3] for examples. We will use the properties listed in the following lemma in our proof.

Lemma 3.1. *Let G be a graph without a 2-factor and (S, T) be a minimum barrier of G . Then the following statements hold.*

- (1) T is independent,
- (2) for every even component C w.r.t. (S, T) , $e_G(T, V(C)) = 0$, and

(3) for every odd component C w.r.t. (S, T) and every $v \in T$, $e_G(v, V(C)) \leq 1$, i.e., either $e_G(v, V(C)) = 0$ or $e_G(v, V(C)) = 1$.

Let (S, T) be a minimum barrier of G . We introduce some necessary notations w.r.t. (S, T) for this paper. Denote

$$\mathcal{C}_k = \{C \in \mathcal{H}_G(S, T) \mid e_G(T, V(C)) = k\}.$$

Then $\mathcal{H}_G(S, T) = \bigcup_{k \geq 0} \mathcal{C}_{2k+1}$ and $h_G(S, T) = |\bigcup_{k \geq 0} \mathcal{C}_{2k+1}|$. For any $v \in T$, let

$$\mathcal{C}_v = \{C \in \mathcal{H}_G(S, T) \mid e_G(v, V(C)) = 1\} \quad \text{and} \quad \mathcal{C}_{1v} = \{C \in \mathcal{C}_1 \mid e_G(v, V(C)) = 1\}.$$

It is clear that $\mathcal{C}_{1v} \subseteq \mathcal{C}_v$. We distinguish \mathcal{C}_{1v} because in the proof of Theorem 1.2, we need pay special attention on vertices $v \in T$ with $\mathcal{C}_{1v} \neq \emptyset$.

Lemma 3.2. *Let G be a graph without a 2-factor and (S, T) be a minimum barrier such that $h_G(S, T)$ is smallest, and let $v \in T$ with $|\mathcal{C}_v| \geq 2$ and $|\mathcal{C}_{1v}| \geq 1$. Then for any vertex w in a component $D \in \mathcal{C}_{1v}$, $e_G(w, V(D) \cup \{v\}) \geq 2$.*

Proof. Suppose on the contrary that there exists a vertex $w \in V(D)$ such that $e_G(w, V(D) \cup \{v\}) \leq 1$. Since $G^* := G[V(D) \cup \{v\}]$ is connected, $e_G(w, V(D) \cup \{v\}) = 1$, which in turn gives that $G^* - w$ is connected. Let $T^* = (T - \{v\}) \cup \{w\}$. We claim that (S, T^*) is a minimum barrier with $h_G(S, T^*) < h_G(S, T)$. This will give a contradiction to the choice of (S, T) .

To see that (S, T^*) is a minimum barrier we calculate $\delta_G(S, T^*)$. Let D_v be the component of $G - (S \cup T^*)$ containing v . Notice that besides v , the component D_v contains also vertices in $D - w$ and all the $d_{G-S}(v) - 1$ odd components $C (\neq D)$ of $G - (S \cup T)$ such that $e_G(v, V(C)) = 1$.

We first show that D_v is an odd component of $G - (S \cup T^*)$. Let v^* be the neighbor of v in D . For each $C \in \mathcal{H}_G(S, T) \cap \mathcal{C}_v$, denote the odd number $e_G(T, V(C))$ by $2k_c + 1$ for some nonnegative integer k_c . If $w = v^*$, then as $e_G(w, V(D) \cup \{v\}) = 1$, we see that D is a single vertex component and $V(G^*) = \{v, w\}$. Then

$$\begin{aligned} e_G(T^*, V(D_v)) &= e_G(T, V(D_v)) - (d_{G-S}(v) - 1) + e_G(v^*, v) \\ &= \sum_{C \in \mathcal{C}_v - \{D\}} e_G(T, V(C)) - (d_{G-S}(v) - 1) + e_G(v^*, v) \\ &= \sum_{C \in \mathcal{C}_v - \{D\}} (2k_c + 1 - 1) + e_G(v^*, v) \quad (\text{noticing that } |\mathcal{C}_v - \{D\}| = d_{G-S}(v) - 1), \end{aligned}$$

which is odd by $e(v^*, v) = d_{G-S}(v^*) = 1$. So D_v is an odd component of $G - (S \cup T^*)$. If $w \neq v^*$, then D has at least two vertices, and

$$\begin{aligned} e_G(T^*, V(D_v)) &= e_G(T, V(D_v)) - d_{G-S}(v) + e_G(w, V(G^*)) \\ &= \sum_{C \in \mathcal{C}_v} e_G(T, V(C)) - d_{G-S}(v) + e_G(w, V(G^*)) \\ &= \sum_{C \in \mathcal{C}_v} (2k_c + 1 - 1) + 1 \quad (\text{noticing that } |\mathcal{C}_v| = d_{G-S}(v)), \end{aligned}$$

which is again odd. Hence, $h_G(S, T^*) = h_G(S, T) - d_{G-S}(v) + 1$. So,

$$\begin{aligned}\delta_G(S, T^*) &= 2|S| - 2|T^*| + \sum_{y \in T^*} d_{G-S}(y) - h_G(S, T^*) \\ &= 2|S| - 2|T| + \sum_{y \in T} d_{G-S}(y) - d_{G-S}(v) + d_{G-S}(w) - (h_G(S, T) - d_{G-S}(v) + 1) \\ &= 2|S| - 2|T| + \sum_{y \in T} d_{G-S}(y) - h_G(S, T) \\ &\leq -2;\end{aligned}$$

by noticing that $d_{G-S}(w) = d_{G^*}(w) = 1$.

As $|S \cup T^*| = |S \cup T|$, (S, T^*) is a minimum barrier. However, as $d_{G-S}(v) = |\mathcal{C}_v| \geq 2$, $h_G(S, T^*) = h_G(S, T) - d_{G-S}(v) + 1 < h_G(S, T)$. This gives a contradiction to the choice of (S, T) . \blacksquare

The following result is a consequence of Lemma 3.2.

Lemma 3.3. *Let G be a graph without a 2-factor and (S, T) be a minimum barrier such that $h_G(S, T)$ is smallest. Then for any $v \in T$ with $|\mathcal{C}_v| \geq 2$ and $D \in \mathcal{C}_{1v}$ (if exists), $|V(D)| \geq 2$.*

4 Proof of the Main Result

Assume, to the contrary, that the n -vertex Δ -critical graph G with $\Delta \geq n/2$ does not have a 2-factor. Then $\Delta \geq 3$ since a 2-critical graph is an odd cycle, which is a 2-factor of G . Since G is Δ -critical, by Vizing's Adjacency Lemma, each vertex of G is adjacent to at least two Δ -vertices and thus $\delta(G) \geq 2$.

By Tutte's 2-factor Theorem (Theorem 3.1), G has a barrier. Let (S, T) be a minimum barrier such that $h_G(S, T)$ is smallest. We use the same notations \mathcal{C}_k , \mathcal{C}_v and \mathcal{C}_{1v} as defined in the previous section.

Claim 4.1. $|T| > |S| + \sum_{k \geq 1} k \cdot |\mathcal{C}_{2k+1}|$.

Proof. Since (S, T) is a barrier,

$$\begin{aligned}\delta_G(S, T) &= 2|S| - 2|T| + \sum_{y \in T} d_{G-S}(y) - h_G(S, T) \\ &= 2|S| - 2|T| + \sum_{y \in T} d_{G-S}(y) - \sum_{k \geq 0} |\mathcal{C}_{2k+1}| < 0.\end{aligned}$$

Let $U = V(G) - (S \cup T)$, by Lemma 3.1 (1) and (2),

$$\sum_{y \in T} d_{G-S}(y) = \sum_{y \in T} e_G(y, U) = e_G(T, U) = \sum_{k \geq 0} (2k+1) |\mathcal{C}_{2k+1}|.$$

Therefore, we have

$$0 > 2|S| - 2|T| + \sum_{k \geq 0} (2k+1)|\mathcal{C}_{2k+1}| - \sum_{k \geq 0} |\mathcal{C}_{2k+1}|,$$

which yields $|T| > |S| + \sum_{k \geq 1} k|\mathcal{C}_{2k+1}|$. \square

Based on the minimum barrier (S, T) , we define two bipartite graphs H^* and H associated with (S, T) as follows. The definitions of H^* and H are fixed hereafter. The bipartite graph H^* is defined as:

$$V(H^*) = X' \cup T \quad \text{where} \quad X' := V(G) - T, \quad \text{and} \quad E(H^*) = E_G(X', T).$$

Notice that the vertices from the even components in $G - (S \cup T)$ (if any) are isolated vertices in H^* by (2) of Lemma 3.1. The bipartite graph H is obtained by performing the following operations to G .

- (1) Remove all even components and all odd components in \mathcal{C}_1 .
- (2) Remove all edges in $G[S]$.
- (3) For a component $C \in \mathcal{C}_{2k+1}$ with $k \geq 1$, contract C into one vertex and then split the resulted vertex into k independent vertices $U^C = \{u_1^C, u_2^C, \dots, u_k^C\}$ such that $d_H(u_1^C) = 3$, and $d_H(u_2^C) = d_H(u_3^C) = \dots = d_H(u_k^C) = 2$. We note that, this operation (3) does nothing to each single vertex component $C \in \mathcal{C}_3$.

Let

$$U^C = \bigcup_{k \geq 1} \left(\bigcup_{C \in \mathcal{C}_{2k+1}} U^C \right) \quad \text{and} \quad X := S \cup U^C.$$

By the constructions, the bipartite graph H satisfies the following properties.

- (1) H is a bipartite graph with partite sets X and T ,
- (2) $|X| = |S| + \sum_{k \geq 1} k|\mathcal{C}_{2k+1}|$, and
- (3) For each $k \geq 1$ and each $C \in \mathcal{C}_{2k+1}$, $d_H(u_1^C) = 3$ and $d_H(u_i^C) = 2$ for each i with $2 \leq i \leq k$.

Note that the construction of H here is a modification of the bipartite graph constructed in [3]. We now introduce some additional notations. Those notations are used heavily in the subsequent proofs.

For each nonnegative integer t , let

$$\mathcal{C}_{\geq(2t+1)} := \bigcup_{k \geq t} \mathcal{C}_{2k+1}.$$

It is clear that $\mathcal{C}_{\geq(2t+1)} \subseteq \mathcal{H}_G(S, T)$. For each $\mathcal{D} \subseteq \mathcal{H}_G(S, T)$, let

$$V(\mathcal{D}) := \bigcup_{C \in \mathcal{D}} V(C), \quad \mathcal{D}^1 = \{C \in \mathcal{D} \mid |V(C)| = 1\} \quad \text{and} \quad \mathcal{D}^2 = \mathcal{D} - \mathcal{D}^1, \quad \text{and} \quad U^{\mathcal{D}} = \left(\bigcup_{C \in \mathcal{D}} U^C \right).$$

For example, we can take $\mathcal{D} = \mathcal{C}_{\geq 3} \subseteq \mathcal{H}_G(S, T)$ in the above definition. Then $V(\mathcal{C}_{\geq 3})$ is the vertex set of all components $C \in \mathcal{H}_G(S, T)$ such that $e_G(T, V(C)) \geq 3$; $\mathcal{C}_{\geq 3}^1$ is the collection of components $C \in \mathcal{C}_{\geq 3}$ such that $|V(C)| = 1$; $\mathcal{C}_{\geq 3}^2$ is the collection of components $C \in \mathcal{C}_{\geq 3}$ such that $|V(C)| \geq 2$; and $U^{\mathcal{C}_{\geq 3}}$ is the set of vertices resulted by splitting each contracted component in \mathcal{C}_{2k+1} into k vertices, for each integer $k \geq 1$.

Denote

$$S' := S \cup V(\mathcal{C}_3^1).$$

Claim 4.2. *Each of the following holds:*

- (1) $d_{H^*}(y) = d_G(y)$ for each $y \in T$;
- (2) $X' \cap X = S' = S \cup V(\mathcal{C}_3^1)$;
- (3) $d_H(x) = d_{H^*}(x)$ for each $x \in S'$;
- (4) $d_H(y) = d_{H^*}(y) - |\mathcal{C}_{1y}|$ for each $y \in T$.

Proof. The statements (1)-(3) are obvious. We only show the last one. By (2) of Lemma 3.1 that for each $y \in T$ and each even component C of $G - (S \cup T)$, $e_G(y, V(C)) = 0$ holds. Thus $d_G(y) = e_G(y, S) + e_G(y, V(\mathcal{C}_{\geq 1})) = e_G(y, S \cup V(\mathcal{C}_{\geq 3})) + e_G(y, V(\mathcal{C}_1)) = d_{H^*}(y)$. By the construction of H , $d_H(y) = e_G(y, S \cup V(\mathcal{C}_{\geq 3})) = d_G(y) - e(y, V(\mathcal{C}_1)) = d_{H^*}(y) - |\mathcal{C}_{1y}|$. \square

In the remaining proof, using Lemma 2.2, we first show that, in the bipartite graph H^* , there is a matching which saturates T . Then by using the relations between H^* and H and Hall's Theorem, we show that, in H , there is a matching which saturates T . The later one gives that $|X| = |S| + \sum_{k \geq 1} k|\mathcal{C}_{2k+1}| \geq |T|$, leading a contradiction to Claim 4.1.

Claim 4.3. *T has no Δ -vertex.*

Proof. Suppose on the contrary that there exists $w \in T$ such that $d_G(w) = \Delta$. Let V_{even} be the vertex set of the even components in $G - (S \cup T)$. Then by $e_G(w, V(\mathcal{C}_{\geq 1})) = |\mathcal{C}_{1w}| + |\mathcal{C}_w - \mathcal{C}_{1w}|$ ((3) of Lemma 3.1) and $|X| = |S| + \sum_{k \geq 1} k|\mathcal{C}_{2k+1}| < |T|$ (Claim 4.1),

$$\begin{aligned} & \frac{1}{2}(|S| + |T| + |V(\mathcal{C}_{\geq 1})| + |V_{\text{even}}|) = \frac{n}{2} \leq \Delta = d_G(w) \leq |S| + |\mathcal{C}_{1w}| + |\mathcal{C}_w - \mathcal{C}_{1w}| \\ & \leq \frac{1}{2}(|S| + 2|\mathcal{C}_{1w}| + |\mathcal{C}_w - \mathcal{C}_{1w}| + (|S| + \sum_{k \geq 1} |\mathcal{C}_{2k+1}|)) \quad (\text{by } \mathcal{C}_w - \mathcal{C}_{1w} \subseteq \bigcup_{k \geq 1} \mathcal{C}_{2k+1}) \\ & < \frac{1}{2}(|S| + 2|\mathcal{C}_{1w}| + |\mathcal{C}_w - \mathcal{C}_{1w}| + |T|). \end{aligned}$$

The above strict inequalities give that

$$\begin{aligned} 2|\mathcal{C}_{1w}| + |\mathcal{C}_w - \mathcal{C}_{1w}| & \geq |V(\mathcal{C}_{\geq 1})| + |V_{\text{even}}| + 1 \\ & \geq (|V(\mathcal{C}_{1w})| + 1) + \sum_{C \in \mathcal{C}_1 - \mathcal{C}_{1w}} |V(C)| + |V(\mathcal{C}_{\geq 3})| + |V_{\text{even}}|. \end{aligned} \tag{1}$$

Since $|V(\mathcal{C}_{\geq 3})| \geq |\mathcal{C}_w - \mathcal{C}_{1w}|$, we have that $2|\mathcal{C}_{1w}| \geq (|V(\mathcal{C}_{1w})| + 1) + \sum_{C \in \mathcal{C}_1 - \mathcal{C}_{1w}} |V(C)| + |V_{\text{even}}|$. If $|\mathcal{C}_{1w}| \geq 2$, then $|V(\mathcal{C}_{1w})| \geq 2|\mathcal{C}_{1w}|$ (Lemma 3.3), showing a contradiction. If $|\mathcal{C}_{1w}| = 0$, then $|V(\mathcal{C}_{1w})| = 0$. So $|\mathcal{C}_{1w}| = 1$. Since $|V(\mathcal{C}_{1w})| \geq |\mathcal{C}_{1w}| = 1$ and $|V(\mathcal{C}_{1w})| + 1 \leq 2$, we get that $|V(\mathcal{C}_{1w})| = 1$ and so (a) $\mathcal{C}_1 = \mathcal{C}_{1w}$ and $|V_{\text{even}}| = 0$. Using $|V(\mathcal{C}_{\geq 3})| \geq |\mathcal{C}_w - \mathcal{C}_{1w}|$ again, under the above facts, inequality (1) becomes (b) $|\mathcal{C}_w - \mathcal{C}_{1w}| = |V(\mathcal{C}_{\geq 3})|$.

Then (a), together with the fact that $|V(\mathcal{C}_{1w})| = 1$ implies that there is exact one single vertex component in \mathcal{C}_1 . By $|V_{\text{even}}| = 0$ in (a), there is no even component in $G - (S \cup T)$. Since $|V(\mathcal{C}_{1w})| = 1$, by Lemma 3.3, $\mathcal{C}_w = \mathcal{C}_{1w}$. By $|\mathcal{C}_w - \mathcal{C}_{1w}| = |V(\mathcal{C}_{\geq 3})|$ in (b), we see that $V(\mathcal{C}_{\geq 3}) = \emptyset$. Let $H^*[X', T]$ and $H[X, T]$ be the two bipartite graphs associated with (S, T) . Then $|X' - X| = |V(\mathcal{C}_1)| = |V(\mathcal{C}_{1w})| = 1$. Combining $|X'| \geq d_{H^*}(w) = d_G(w) = \Delta \geq n/2$, we have $|X| \geq n/2 - 1$. As $|T| > |X|$ (Claim 4.1) and $|T| + |X'| = |T| + |X| + 1 = n$, we get $|T| = |X'| = n/2$ and $|X| = n/2 - 1 \leq \Delta - 1$. Hence, for any $y \in T \cap V_\Delta$, y is adjacent to the unique vertex in $\mathcal{C}_{1w} = \mathcal{C}_1$. As $e_G(T, V(\mathcal{C}_{1w})) = 1$, $T \cap V_\Delta = \{w\}$. That is, w is the unique Δ -vertex in T . Applying Lemma 2.4 with $\delta_0 = 1$ on H^* , we see that for every edge $xy \in E(H^*)$ with $x \in X'$ and $y \in T$, the relation $d_{H^*}(y) \geq d_{H^*}(x)$ holds. As $d_{H^*}(y) = d_G(y) \geq 2$, T has no isolated vertices. Hence in H^* , there is a matching M which saturates T . Since $|X'| = |T|$, M is a perfect matching of H^* . Let w^* be the vertex to which w is adjacent in \mathcal{C}_1 . Then $V(M)$ contains w^* and $d_{H^*}(w^*) = 1$ by noticing that $d_{H^*}(w^*) = e_G(w^*, T) = e_G(T, V(\mathcal{C}_{1w})) = 1$. As for any $y \in T$, $d_{H^*}(y) = d_G(y) \geq 2$, $d_{H^*}(w) > d_{H^*}(w^*)$. For any $xy \in E(H^*) - \{ww^*\}$ with $x \in X'$ and $y \in T$, $d_{H^*}(y) \geq d_{H^*}(x)$. Hence,

$$\begin{aligned} e_{H^*}(X', T) &= \sum_{\substack{xy \in M - \{ww^*\} \\ x \in X', y \in T}} d_{H^*}(y) + d_{H^*}(w) \\ &> \sum_{\substack{xy \in M - \{ww^*\} \\ x \in X', y \in T}} d_{H^*}(x) + d_{H^*}(w^*) = e_{H^*}(X', T), \end{aligned}$$

showing a contradiction. \square

For a vertex $x \in X' = V(G) - T$, define σ_x as the number of non Δ -degree neighbors of x in X' and let

$$S_1 = \{x \in S' = S \cup V(\mathcal{C}_3^1) \mid \sigma_x \geq 1\} \quad \text{and} \quad S_0 = S' - S_1 = \{x \in S' \mid \sigma_x = 0\}.$$

Following the definitions of S_0 and S_1 , $N_G(x) \cap S_0 = \emptyset$ for any non Δ -degree vertex $x \in X'$.

As T has no Δ -vertex, applying Lemma 2.4 with $\delta_0 = 0$, for each edge $xy \in E(H^*)$ with $x \in X'$ and $y \in T$, we have the following claim.

Claim 4.4.

$$d_{H^*}(y) \geq \begin{cases} d_{H^*}(x) + 2, & \text{if } x \in S_1; \\ d_{H^*}(x) + 1, & \text{otherwise.} \end{cases}$$

Moreover, H^* has a matching which saturates T .

Claim 4.5. Let $C \in \mathcal{C}_{\geq 1}$ and $x \in V(C) \cap V_\Delta$. If $e_G(x, T) \leq 1$, then $|V(C)| > \frac{1}{2}|V(\mathcal{C}_{\geq 1})|$.

Proof. Suppose on the contrary that $|V(C)| \leq \frac{1}{2}|V(\mathcal{C}_{\geq 1})|$, that is, $|V(C)| \leq |V(\mathcal{C}_{\geq 1})| - |V(C)|$. Then since $e_G(x, T) \leq 1$,

$$n/2 \leq \Delta = d_G(x) \leq |S| + |V(C)| - 1 + e_G(x, T) \leq |S| + |V(C)| \leq |S| + |V(\mathcal{C}_{\geq 1})| - |V(C)|.$$

As $|T| > |S|$,

$$n \leq 2|S| + |V(C)| + |V(\mathcal{C}_{\geq 1})| - |V(C)| < |S| + |T| + |V(C)| + |V(\mathcal{C}_{\geq 1})| - |V(C)| \leq n,$$

showing a contradiction. \square

For each $y \in T$, we have $d_{H^*}(y) = d_G(y) \geq 2$. The following claim gives a property when $d_{H^*}(y) = 2$.

Claim 4.6. *Let $y \in T$ be a vertex. If $d_{H^*}(y) = 2$, then there exists $x \in N_{H^*}(y) \cap S$ such that $d_{H^*}(x) = 1$.*

Proof. By Vizing's Adjacency Lemma, each vertex of G is adjacent to at least two Δ -vertices. Since $d_{H^*}(y) = d_G(y) = 2$, the two neighbors of y are Δ -vertices. By Claim 4.4, $2 = d_{H^*}(y) \geq d_{H^*}(x) + 1$, so each of the two neighbors of y has degree 1 in H^* (and thus has exact one neighbor in T). If $N_{H^*}(y) \subseteq V(\mathcal{C}_{\geq 1})$, then by (3) of Lemma 3.1, each of the vertex in $N_{H^*}(y)$ is contained in a distinct component in $\mathcal{C}_{\geq 1}$. However, by Claim 4.5, there exists at most one component $C \in \mathcal{C}_{\geq 1}$ such that it contains a Δ -vertex and the Δ -vertex has degree exact 1 in H^* . Hence $N_{H^*}(y) \cap S \neq \emptyset$. Let $x \in N_{H^*}(y) \cap S$. Then x is the desired vertex. \square

Claim 4.7. *We may assume that $\mathcal{C}_1 \neq \emptyset$.*

Proof. Suppose on the contrary that $\mathcal{C}_1 = \emptyset$. By Claim 4.2, for each $x \in S$, $d_H(x) = d_{H^*}(x)$ and for any $y \in T$, $d_H(y) = d_{H^*}(y)$. Applying Claim 4.6, if there exists $y \in T$ such that $d_H(y) = 2$, then y has a neighbor of degree 1 in S . Let $X_1 = \{x \in X \mid d_H(x) = 1\}$ and $T_1 = N_H(X_1)$, and let $H'[X - X_1, T - T_1] := H[(T - T_1) \cup (X - X_1)]$. Then for any $y \in T - T_1$, $d_{H'}(y) \geq 3$. We then claim that for each edge $xy \in E(H')$ with $x \in X - X_1$ and $y \in T - T_1$, $d_{H'}(y) \geq d_{H'}(x)$ holds. If $x \in S$, then by Claim 4.4, $d_{H^*}(x) \leq d_{H^*}(y)$. Hence $d_{H'}(x) \leq d_H(x) = d_{H^*}(x) \leq d_{H^*}(y) = d_H(y) = d_{H'}(y)$. If $x \in U^c$, then $d_{H'}(x) \leq d_H(x) \leq 3$ by the construction of H . Hence $d_{H'}(x) \leq 3 \leq d_{H'}(y)$. Since $N_H(X_1) = T_1$, that T has no isolated vertices in H implies that $T - T_1$ has no isolated vertices in H' . By Lemma 2.2, H' has a matching which saturates $T - T_1$. By Lemma 2.3, H has a matching which saturates T . This gives a contradiction to Claim 4.1. \square

For each $C \in \mathcal{C}_1$, by (3) of Lemma 3.1, there is a unique vertex $y_c \in T$ adjacent to a unique vertex x_c on it. We call x_c and y_c the partners of each other. We divide the components in \mathcal{C}_1 into two subgroups \mathcal{C}_{11} and \mathcal{C}_{12} in order to consider the degrees of the partner vertices in T :

$$\mathcal{C}_{11} = \{C \in \mathcal{C}_1 \mid e_G(y_c, \mathcal{C}_1) = 1\} \quad \text{and} \quad \mathcal{C}_{12} = \{C \in \mathcal{C}_1 \mid e_G(y_c, \mathcal{C}_1) \geq 2\}.$$

By the definition of \mathcal{C}_{12} , it is clear that if $\mathcal{C}_{12} \neq \emptyset$, then $|\mathcal{C}_{12}| \geq 2$. Also, by Lemma 3.3, for each $C \in \mathcal{C}_{12}$, $|V(C)| \geq 2$.

Furthermore, we divide the components in \mathcal{C}_{11} into two groups as follows.

$$\mathcal{C}_{11}^1 = \{C \in \mathcal{C}_{11} \mid |V(C)| = 1\} \quad \text{and} \quad \mathcal{C}_{11}^2 = \{C \in \mathcal{C}_{11} \mid |V(C)| \geq 2\}.$$

Corresponding to the partition of \mathcal{C}_1 , we partition vertices in T into subgroups, as follows.

$$\begin{aligned} T_1^1 &= \{y \in T \mid e_G(y, V(\mathcal{C}_{11}^1)) = 1\}, & T_1^2 &= \{y \in T \mid e_G(y, V(\mathcal{C}_{11}^2)) = 1\}; \\ T_0 &= \{y \in T \mid e_G(y, V(\mathcal{C}_1)) = 0\}, & \text{and} & \quad T_2 = \{y \in T \mid e_G(y, V(\mathcal{C}_{12})) \geq 2\}. \end{aligned}$$

Notice that for a vertex $y \in T$, if $e_G(y, V(\mathcal{C}_1)) \geq 2$, then $e_G(y, V(\mathcal{C}_1)) = e_G(y, V(\mathcal{C}_{12}))$. Hence,

$$e_G(y, V(\mathcal{C}_1)) = 1 \quad \text{for each} \quad y \in T_1^1 \cup T_1^2. \quad (2)$$

Since each $C \in \mathcal{C}_{11}^1$ satisfies $|V(C)| = 1$, by Lemma 3.3,

$$e_G(y, V(\mathcal{C}_{\geq 1}^1)) = 1 \quad \text{for each} \quad y \in T_1^1. \quad (3)$$

Let $m_{11} := |\mathcal{C}_{11}^1|$, $m_{12} := |\mathcal{C}_{11}^2|$, $m_2 := |\mathcal{C}_{12}|$, and $m_3 := |\mathcal{C}_{\geq 3}^2|$.

Claim 4.8. *We may assume that none vertices in $V(\mathcal{C}_{11}^1)$ is a Δ -vertex.*

Proof. Suppose on the contrary and let $x_c \in V(\mathcal{C}_{11}^1) \cap V_\Delta$. Since $e_G(x_c, T) = 1$ and $e_G(x_c, V(G) - S - T) = 0$, we have $e_G(x_c, S) = \Delta - 1 \geq n/2 - 1$. This indicates that $|S| \geq n/2 - 1$. Combining $|T| > |S| + \sum_{k \geq 1} k|\mathcal{C}_{2k+1}|$ (Claim 4.1) and $|S| + |T| < |X'| + |T| = n$ (noticing that $|S| < |X'|$ by $1 = |\{x_c\}| \leq |V(\mathcal{C}_{11}^1)|$ and $V(\mathcal{C}_{11}^1) \cup S \subseteq X'$), we have $|T| = n/2 = |S| + 1$. We consider the bipartite graph $H^*[X', T]$ associated with (S, T) . As $|V(G)| = n$ and $|T| = n/2$, $|X'| = |T|$. By Claim 4.4, H^* has a matching M which saturates T . Since $|T| = |X'|$, M is a perfect matching. Since $d_{H^*}(x_c) = 1$, T has a unique neighbor, say y_c of x_c . Then $x_c y_c \in M$. Because $d_{H^*}(y_c) = d_G(y_c) \geq 2$, $d_{H^*}(y_c) > d_{H^*}(x_c)$. By Claim 4.4, $d_{H^*}(y) \geq d_{H^*}(x)$ for each $xy \in E(H^*) - \{x_c y_c\}$ with $x \in X'$ and $y \in T$. Hence,

$$\begin{aligned} e_{H^*}(X', T) &= \sum_{\substack{xy \in M - \{x_c y_c\} \\ x \in X', y \in T}} d_{H^*}(y) + d_{H^*}(y_c) \\ &> \sum_{\substack{xy \in M - \{x_c y_c\} \\ x \in X', y \in T}} d_{H^*}(x) + d_{H^*}(x_c) = e_{H^*}(X', T), \end{aligned}$$

showing a contradiction. □

By the definition, for each $C \in \mathcal{C}_{\geq 3}^2 \cup \mathcal{C}_{11}^2 \cup \mathcal{C}_{12}$, we have $|V(C)| \geq 2$ holds. Thus

$$\begin{aligned} n &\geq |S'| + |T| + |V(\mathcal{C}_{11}^1)| + |V(\mathcal{C}_{11}^2)| + |V(\mathcal{C}_{12})| + |V(\mathcal{C}_{\geq 3}^2)| \\ &\geq |S'| + |T| + m_{11} + 2m_{12} + 2m_2 + 2m_3, \end{aligned} \quad (4)$$

where $S' = S \cup V(\mathcal{C}_3^1)$ is defined previously.

Claim 4.9. *Let $xy \in E(H^*)$ be an edge with $x \in V(C) \subseteq V(\mathcal{C}_{\geq 5}^1)$ and $y \in T$, and let u_c be a vertex in U^C which is adjacent to y in H . Then $d_{H^*}(y) \geq d_H(u_c) + 3$.*

Proof. Let $V(C) = \{x\}$. Then $d_{H^*}(x) \geq 5$ as $C \in \mathcal{C}_{\geq 5}$. By Claim 4.4, $d_{H^*}(y) \geq 6$. Recall that in U^C , $d_H(u_1^C) = 3$ and $d_H(u_i^C) = 2$ for $i \geq 2$, so $d_{H^*}(y) \geq 6 \geq d_H(u_c) + 3$. \square

For each vertex $y \in T_1^1 \cup T_1^2 \cup T_2$, $|\mathcal{C}_{1y}| \geq 1$. So $d_H(y) = d_{H^*}(y) - |\mathcal{C}_{1y}| < d_{H^*}(y) = d_G(y)$. In order to find a matching saturating T in H , in the following three claims, we show that y still has enough neighbors remained in $V(G) - T - V(\mathcal{C}_1) = S \cup V(\mathcal{C}_{\geq 3})$.

Claim 4.10. *If $T_1^1 \neq \emptyset$, then for each $y \in T_1^1$,*

$$|N_G(y) \cap V_\Delta \cap S| \geq |S_0| + (m_{11} + 1)/2 + m_{12} + m_2 + m_3.$$

Proof. Let $x \in V(\mathcal{C}_{11}^1)$ such that $xy \in E(G)$. Since $y \in T_1^1$, using (3) that $e(y, V(\mathcal{C}_{\geq 1})) = 1 = e(y, V(\mathcal{C}_{11}^1))$, $N_G(y) \cap V(\mathcal{C}_{\geq 1}) = \{x\}$. By Claim 4.8, x is not a Δ -vertex. Thus $N_G(y) \cap V_\Delta \subseteq S$. So we only need to show $|N_G(y) \cap V_\Delta| \geq |S_0| + (m_{11} + 1)/2 + m_{12} + m_2 + m_3$. Recall $S_0 = \{x \in S' \mid \sigma_x = 0\}$ is the set of vertices in $S' = S_0 \cup S_1$ only adjacent to Δ -vertices in $X' = V(G) - T$, so $N_G(x) \cap S_0 = \emptyset$. Hence $d_G(x) \leq |N_G(x) \cap T| + |N_G(x) \cap S_1| \leq |S_1| + 1$. By Vizing's Adjacency Lemma, y is adjacent to at least $\Delta - d_G(x) + 1$ Δ -vertices in G . Simple calculation shows that

$$\begin{aligned} \Delta - d_G(x) + 1 &\geq n/2 - |S_1| - 1 + 1 \\ &\geq \frac{1}{2}(|S'| + |T| + m_{11} + 2m_{12} + 2m_2 + 2m_3) - |S_1| \quad (\text{by inequality (4)}) \\ &\geq |S_0| + (m_{11} + 1)/2 + m_{12} + m_2 + m_3, \end{aligned}$$

where the last inequality is obtained by using the facts that $|S'| = |S_1| + |S_0|$ and $|S'| = |S| + |V(\mathcal{C}_3^1)| = |S| + |\mathcal{C}_3^1| \leq |S| + \sum_{k \geq 1} |\mathcal{C}_{2k+1}| < |T|$ by Claim 4.1. \square

If $\mathcal{C}_{11}^2 \neq \emptyset$, let $C_{\max}^1 \in \mathcal{C}_{11}^2$ be a component such that $|V(C_{\max}^1)| = \max\{|V(C)| \mid C \in \mathcal{C}_{11}^2\}$. Then by Claim 4.5, if $V(\mathcal{C}_{11}) \cap V_\Delta \neq \emptyset$, then $V(\mathcal{C}_{11}) \cap V_\Delta = V(C_{\max}^1) \cap V_\Delta$ (since $\mathcal{C}_{11} = \mathcal{C}_{11}^1 \cup \mathcal{C}_{11}^2$ and $V(\mathcal{C}_{11}^1) \cap V_\Delta = \emptyset$ by Claim 4.8).

Claim 4.11. *If $T_1^2 \neq \emptyset$, then for each $y \in T_1^2$,*

$$|N_G(y) \cap V_\Delta \cap (S \cup V(\mathcal{C}_{\geq 3}))| \geq \begin{cases} |S_0| + (m_{11} + 1)/2 + m_{12} + m_2 + m_3 - 1, & \text{if } x \notin V(C_{\max}^1); \\ 1, & \text{if } x \in V(C_{\max}^1); \end{cases}$$

where x is the neighbor of y in \mathcal{C}_{11}^2 .

Proof. Suppose first that $C \neq C_{\max}^1$. Since y is adjacent to exactly one component C in \mathcal{C}_{11}^2 (by (2)) and C contains no Δ -vertex, $N_G(y) \cap V_\Delta \subseteq S \cup V(\mathcal{C}_{\geq 3})$. So we only need to show $|N_G(y) \cap V_\Delta| \geq |S_0| + (m_{11} + 1)/2 + m_{12} + m_2 + m_3 - 1$. Again, as $x \notin V_\Delta$, $N_G(x) \cap S_0 = \emptyset$. Hence $d_G(x) \leq |N_G(x) \cap T| + |N_G(x) \cap S_1| + |V(C)| - 1 \leq$

$|S_1| + |V(C)|$. By Vizing's Adjacency Lemma, y is adjacent to at least $\Delta - d_G(x) + 1$ Δ -vertices in G . Since $n \geq |S'| + |T| + m_{11} + |V(C)| + |V(C_{\max}^1)| + 2(m_{12} - 2) + 2m_2 + 2m_3$,

$$\begin{aligned} \Delta - d_G(x) + 1 &\geq n/2 - (|S_1| + |V(C)|) + 1 \\ &\geq \frac{1}{2}(|S'| + |T| + m_{11} + |V(C)| + |V(C_{\max}^1)| + 2(m_{12} - 2) + 2m_2 + 2m_3) - |S_1| - |V(C)| + 1 \\ &\geq |S_0| + (m_{11} + 1)/2 + m_{12} + m_2 + m_3 - 1. \end{aligned}$$

Suppose now that $C = C_{\max}^1$. As $d_G(y) \geq 2$, and $e_G(y, V(C_1)) = 1$ by (2), the other neighbor of y is contained in $S \cup V(\mathcal{C}_{\geq 3})$. \square

If C_{\max}^1 exists, let y_s (for y_{special}) be the unique vertex in T such that $e_G(y_s, V(C_{\max}^1)) = 1$.

Claim 4.12. *If $T_2 \neq \emptyset$, then for each $y \in T_2$,*

$$|N_G(y) \cap V_{\Delta} \cap (S \cup V(\mathcal{C}_{\geq 3}))| \geq |S_0| + m_{11}/2 + m_{12} + m_2 + m_3 - 1.$$

Proof. If $T_2 \neq \emptyset$, then by the definition, $\mathcal{C}_{12} \neq \emptyset$, giving that $m_2 \geq 2$. Let C_{\max}^2 be a component with largest cardinality in \mathcal{C}_{12} . Let $C_1 \in \mathcal{C}_{12} \cap \mathcal{C}_{1y} - \{C_{\max}^2\}$ and x be the neighbor of y on C_1 . By Claim 4.5, if C_{\max}^2 contains a Δ -vertex, it is the only component in \mathcal{C}_1 which contains a Δ -vertex. Thus, $N_G(y) \cap V_{\Delta} \subseteq S \cup V(\mathcal{C}_{\geq 3}) \cup V(C_{\max}^2)$. So it suffices to show that $|N_G(y) \cap V_{\Delta}| - |N_G(y) \cap V(C_{\max}^2) \cap V_{\Delta}| \geq |S_0| + m_{11}/2 + m_{12} + m_2 + m_3 - 1$. Again, as $x \notin V_{\Delta}$, $N_G(x) \cap S_0 = \emptyset$. Hence $d_G(x) \leq |N_G(x) \cap T| + |N_G(x) \cap S_1| + |V(C)| - 1 \leq |S_1| + |V(C)|$. By Vizing's Adjacency Lemma, y is adjacent to at least $\Delta - d_G(x) + 1$ Δ -vertices in G . So y has at least $\Delta - d_G(x) + 1$ Δ -degree neighbors in $S \cup V(\mathcal{C}_{\geq 3})$ if $|N_G(y) \cap V(C_{\max}^2) \cap V_{\Delta}| = 0$; and y has at least $\Delta - d_G(x)$ Δ -degree neighbors in $S \cup V(\mathcal{C}_{\geq 3})$ if $|N_G(y) \cap V(C_{\max}^2) \cap V_{\Delta}| = 1$.

If $|N_G(y) \cap V(C_{\max}^2) \cap V_{\Delta}| = 0$, we get that

$$\begin{aligned} \Delta - d_G(x) + 1 &\geq n/2 - (|S_1| + |V(C_1)|) + 1 \\ &\geq \frac{1}{2}(|S'| + |T| + m_{11} + 2m_{12} + |V(C_1)| + |V(C_{\max}^2)| + 2(m_2 - 2) + 2m_3) - |S_1| - |V(C_1)| + 1 \\ &\geq |S_0| + (m_{11} + 1)/2 + m_{12} + m_2 + m_3 - 1. \end{aligned}$$

If $|N_G(y) \cap V(C_{\max}^2) \cap V_{\Delta}| = 1$, then C_{\max}^2 contains a Δ -vertex x with $e_G(x, T) = 1$, and thus $|V(C_{\max}^2)| > |V(C_1)|$ by Claim 4.5. Also since $|S'| < |T|$, we get

$$\begin{aligned} \Delta - d_G(x) &\geq n/2 - (|S_1| + |V(C_1)|) \\ &\geq \frac{1}{2}(|S'| + |T| + m_{11} + 2m_{12} + |V(C_1)| + |V(C_{\max}^2)| + 2(m_2 - 2) + 2m_3) - |S_1| - |V(C_1)| \\ &\geq |S_0| + m_{11}/2 + m_{12} + m_2 + m_3 - 1. \end{aligned}$$

\square

Claim 4.13. *In the bipartite graph $H[X, T]$, T has no isolated vertices.*

Proof. Let $y \in T$ be a vertex. If $|\mathcal{C}_1| \leq 1$, then $d_H(y) \geq d_{H^*}(y) - 1 = d_G(y) - 1 \geq 1$. So assume $|\mathcal{C}_1| \geq 2$. If $y \in T_0$, then $d_H(y) = d_{H^*}(y) = d_G(y) \geq 2$. For each $y \in T_1^1 \cup T_1^2 \cup T_2$, either $d_H(y) \geq m_{11}/2 + m_{12} + m_2 - 1$ or $d_H(y) \geq 1$ by claims 4.10-4.12. Since $m_{11} + m_{12} + m_2 = |\mathcal{C}_1| \geq 2$, $d_H(y) \geq 1$. \square

Claim 4.14. *Let $xy \in E(H)$ be an edge with $x \in X$ and $y \in T$. Then each of the following holds:*

- (1) $d_H(y) + |\mathcal{C}_{1y}| \geq d_H(x) + 2$ if $x \in S_1$;
- (2) $d_H(y) + |\mathcal{C}_{1y}| \geq d_H(x) + 1$ if $x \in S_0$;
- (3) $d_H(y) + |\mathcal{C}_{1y}| \geq d_H(x) + 3$ if $x \in U^{\mathcal{C}_1^1 \geq 5}$;
- (4) For each $x \in U^{\mathcal{C}_2^2 \geq 3}$, either $d_H(y) + |\mathcal{C}_{1y}| \geq d_H(x)$ or $d_H(y) + |\mathcal{C}_{1y}| = 2$ and $d_H(x) = 3$. In the later case, there exists $x \in S$ such that $xy \in E(H)$ and $d_H(x) = 1$.

Proof. As $d_H(y) + |\mathcal{C}_{1y}| = d_{H^*}(y)$ and $d_H(x) = d_{H^*}(x)$ for all $x \in S' = X \cap X'$, (1)-(2) follow Claim 4.4. By Claim 4.9, we get (3). For each $y \in T$, $d_{H^*}(y) = d_H(y) + |\mathcal{C}_{1y}| = d_G(y) \geq 2$, and for each $x \in U^{\mathcal{C}_2^2 \geq 3}$, according to the construction of H , either $d_H(x) = 2$ or $d_H(x) = 3$. If $d_{H^*}(y) \geq 3$, the first part of (4) holds. If $d_{H^*}(y) = 2$, then the second part of (4) follows. The existence of the vertex $x \in S$ such that $xy \in E(H)$ and $d_H(x) = 1$ is guaranteed by Claim 4.6. \square

Let $y \in T$ be a vertex of degree 2 in H^* . By Claim 4.6, y has a neighbor x in S which has degree 1 in H^* . As $S \subseteq X \cap X'$, y has a neighbor x of degree 1 also in H . Applying Lemma 2.3, to show that H has a matching which saturates T , we may assume that for any vertex $y \in T$, $d_{H^*}(y) \geq 3$ holds. By Claim 4.14, the assumption indicates that

$$d_H(y) + |\mathcal{C}_{1y}| \geq 3 \quad \text{and} \quad d_H(y) + |\mathcal{C}_{1y}| \geq d_H(x) \text{ for every edge } xy \in E(H). \quad (5)$$

Claim 4.15. *H has a matching which saturates T .*

Proof. Suppose not. Then by Hall's Theorem, there is a nonempty set $A \subseteq T$ such that $|N_H(A)| < |A|$. We choose A such that it has the minimum cardinality and satisfies $|N_H(A)| < |A|$. Let $B := N_H(A)$ and $H' := H[A \cup B]$. We claim that, in H' , there is a matching which saturates B . Suppose on the contrary. Then by Hall's Theorem again, there is a nonempty subset $B' \subseteq B$ such that $|N_{H'}(B')| < |B'|$. Since $B' \subseteq B = N_H(A) \neq \emptyset$ (T has no isolated vertices by Claim 4.13), $N_{H'}(B') \neq \emptyset$. Let $A' = A - N_{H'}(B')$. As $|A| > |B| \geq |N_{H'}(B')| > 0$, $0 < |A'| < |A|$. On the other hand, we have $N_{H'}(A') = N_H(A') = B - B'$. However, $|A'| = |A| - |N_{H'}(B')| > |B| - |N_{H'}(B')| > |B| - |B'| = |B - B'| = |N_H(A')|$, showing a contradiction to the choice of A .

In H' , let M be a matching which saturates B . We consider three cases below.

Case 1. $A \subseteq T_0$.

In this case, all vertices $y \in A$ has $|\mathcal{C}_{1y}| = 0$. By Claim 4.14, $d_{H'}(y) = d_H(y) \geq d_{H'}(x)$ for every edge $xy \in E(H')$. As $|A| > |B|$ and M saturates B , $A - V(M) \neq \emptyset$. Let $y^* \in A - V(M)$. Then $d_H(y^*) \geq 3$ by (5). Then we get

$$\begin{aligned} e_H(A, B) &= \sum_{\substack{xy \in M \\ x \in B, y \in A}} d_{H'}(x) \leq \sum_{\substack{xy \in M \\ x \in B, y \in A}} d_H(x) \\ &\leq \sum_{\substack{xy \in M \\ x \in B, y \in A}} (d_H(y) + |\mathcal{C}_{1y}|) \quad (\text{by (5)}) \\ &< \sum_{\substack{xy \in M \\ x \in B, y \in A}} d_H(y) + d_H(y^*) \leq e_H(A, B) \quad (|\mathcal{C}_{1y}| = 0 \text{ for } y \in A \text{ and } d_H(y^*) \geq 3), \end{aligned}$$

showing a contradiction.

Case 2. $A \cap (T_1^1 \cup T_1^2 \cup T_2) = A \cap T_1^2 = \{y_s\}$.

Note that in this case, $y_s \in T_1^2$ and thus $|\mathcal{C}_{1y_s}| = e_G(y_s, V(\mathcal{C}_1)) = 1$ by (2). Since $A \cap (T_1^1 \cup T_1^2 \cup T_2) = \{y_s\}$, for each $y \in A - \{y_s\}$, $|\mathcal{C}_{1y}| = 0$. As $|A| > |B|$ and M saturates B , $A - V(M) \neq \emptyset$. Let $y^* \in A - V(M)$. Following (5), we have that $d_{H^*}(y) + |\mathcal{C}_{1y^*}| \geq 3$. If $y^* \neq y_s$, $d_H(y^*) = d_{H^*}(y^*) + |\mathcal{C}_{1y^*}| \geq 3$; and if $y^* = y_s$ then $|\mathcal{C}_{1y^*}| = 1$, so $d_H(y^*) \geq 2$. We may assume that $y_s \in A \cap V(M)$, for otherwise, we can get a contradiction by the same argument as in Case 1. Hence,

$$\begin{aligned} e_H(A, B) &= \sum_{\substack{xy \in M \\ x \in B, y \in A}} d_{H'}(x) \leq \sum_{\substack{xy \in M \\ x \in B, y \in A}} d_H(x) \\ &\leq \sum_{\substack{xy \in M \\ x \in B, y \in A - \{y_s\}}} (d_H(y) + |\mathcal{C}_{1y}|) + (d_H(y_s) + 1) \quad (\text{by (5)}) \\ &\leq \sum_{\substack{xy \in M \\ x \in B, y \in A}} d_H(y) + 1 + (d_H(y^*) - 3) \leq e_H(A, B) - 2 \quad (|\mathcal{C}_{1y}| = 0 \text{ for } y \in A - \{y_s\} \text{ and } d_H(y^*) \geq 3), \end{aligned}$$

giving a contradiction.

Case 3. $A \cap (T_1^1 \cup T_1^2 \cup T_2) - \{y_s\} \neq \emptyset$.

Let $y' \in A \cap (T_1^1 \cup T_1^2 \cup T_2) - \{y_s\}$ such that $|N_G(y')| = \max_{y \in A \cap (T_1^1 \cup T_1^2 \cup T_2) - \{y_s\}} |N_G(y)|$. Denote $B_1 := N_H(y')$ and $\overline{B_1} := B - B_1$. Then $V(M) \cap B = B = B_1 \cup \overline{B_1}$. Since $y' \neq y_s$, we have $|B_1| = |N_G(y')| - |\mathcal{C}_{1y'}| \geq |N_G(y') \cap V_\Delta \cap (S \cup V(\mathcal{C}_{\geq 3}))|$. So

$$|B_1| \geq |N_G(y') \cap V_\Delta \cap (S \cup V(\mathcal{C}_{\geq 3}))| \geq \begin{cases} |S_0| + (m_{11} + 1)/2 + m_{12} + m_2 + m_3, & \text{if } y' \in T_1^1 \text{ (Claim 4.10);} \\ |S_0| + (m_{11} + 1)/2 + m_{12} + m_2 + m_3 - 1, & \text{if } y' \in T_1^2 \text{ (Claim 4.11);} \\ |S_0| + m_{11}/2 + m_{12} + m_2 + m_3 - 1, & \text{if } y' \in T_2 \text{ (Claim 4.12).} \end{cases}$$

In notching that if $m_2 > 0$ then $m_2 \geq 2$, using the above lower bounds on $|B_1|$, we claim the following.

$$|B_1| \geq \begin{cases} |S_0| + (m_{11} + 1)/2 + m_3, & \text{(a) if } m_{12} = 0, 1 \text{ and } m_2 = 0; \\ |S_0| + m_{11}/2 + m_2 + m_3 - 1, & \text{(b) if } m_{12} = 0, m_2 \geq 2; \\ |S_0| + (m_{11} + 1)/2 + m_{12} + m_3 - 1, & \text{(c) if } m_{12} \geq 2 \text{ and } m_2 = 0; \\ |S_0| + m_{11}/2 + m_{12} + m_2 + m_3 - 1, & \text{(d) if } m_{12} \geq 1 \text{ and } m_2 \geq 2. \end{cases} \quad (6)$$

We verify (a). Notice that when $m_{12} = 0$ and $m_2 = 0$, $T_1^2 = T_2 = \emptyset$, which implies $y' \in T_1^1$. By Claim 4.10 we get $|B_1| \geq |S_0| + (m_{11} + 1)/2 + m_3$. When $m_{12} = 1$ and $m_2 = 0$, $y' \in T_1^1 \cup T_1^2$. Then $|B_1| \geq \min\{|S_0| + (m_{11} + 1)/2 + 1 + m_3, |S_0| + (m_{11} + 1)/2 + 1 + m_3 - 1\} = |S_0| + (m_{11} + 1)/2 + m_3$. Similarly, we can verify (b), (c) and (d).

By Claim 4.14 and (5), for each edge $xy \in E(H')$ with $x \in B$ and $y \in A$, we have three cases:

- (i) $d_H(y) + |\mathcal{C}_{1y}| \geq 3 \geq d_H(x)$ if $x \in B_1 \cap U^{\mathcal{C}_{\geq 3}^2}$, where $|B_1 \cap U^{\mathcal{C}_{\geq 3}^2}| = |N_H(y') \cap U^{\mathcal{C}_{\geq 3}^2}| = |N_G(y') \cap V(\mathcal{C}_{\geq 3}^2)| \leq |\mathcal{C}_{\geq 3}^2| = m_3$ by (3) of Lemma 3.1;
- (ii) $d_H(y) + |\mathcal{C}_{1y}| \geq d_H(x) + 1$ if $x \in B_1 \cap S_0$; and
- (iii) $d_H(y) + |\mathcal{C}_{1y}| \geq d_H(x) + 2$ if $x \in B_1 - U^{\mathcal{C}_{\geq 3}^2} - S_0$.

As $|A| > |B|$ and M saturates B , $A - V(M) \neq \emptyset$. Let $y^* \in A - V(M)$. Then $d_H(y^*) + |\mathcal{C}_{1y^*}| \geq 3$ by (5). Hence

$$\begin{aligned} e_H(A, B) &= \sum_{\substack{xy \in M \\ x \in B, y \in A}} d_{H'}(x) \leq \sum_{\substack{xy \in M \\ x \in B, y \in A}} d_H(x) \quad (7) \\ &= \sum_{\substack{xy \in M \\ x \in B_1, y \in A}} d_H(x) + \sum_{\substack{xy \in M \\ x \in \overline{B_1}, y \in A}} d_H(x) \\ &\leq \sum_{\substack{xy \in M \\ x \in B_1 \cap S_0, y \in A}} (d_H(y) + |\mathcal{C}_{1y}| - 1) + \sum_{\substack{xy \in M \\ x \in B_1 \cap U^{\mathcal{C}_{\geq 3}^2}, y \in A}} (d_H(y) + |\mathcal{C}_{1y}|) + \\ &\quad \sum_{\substack{xy \in M \\ x \in B_1 - (S_0 \cup U^{\mathcal{C}_{\geq 3}^2}), y \in A}} (d_H(y) + |\mathcal{C}_{1y}| - 2) + \sum_{\substack{xy \in M \\ x \in \overline{B_1}, y \in A}} (d_H(y) + |\mathcal{C}_{1y}|) \\ &\leq \sum_{\substack{xy \in M \\ x \in B, y \in A}} d_H(y) - |B_1 \cap S_0| - 2|B_1 - (S_0 \cup U^{\mathcal{C}_{\geq 3}^2})| + \sum_{y \in A \cap V(M)} |\mathcal{C}_{1y}| \\ &< \sum_{\substack{xy \in M \\ x \in B, y \in A}} d_H(y) - |B_1 \cap S_0| - 2|B_1 - (S_0 \cup U^{\mathcal{C}_{\geq 3}^2})| + \sum_{y \in A \cap V(M) \cup \{y^*\}} |\mathcal{C}_{1y}| + d_H(y^*). \quad (8) \end{aligned}$$

As $|B_1 \cap S_0| \leq |S_0|$, and $|B_1 \cap U^{\mathcal{C}_{\geq 3}^2}| \leq m_3$, $2|B_1 - (S_0 \cup U^{\mathcal{C}_{\geq 3}^2})| \geq 2(|B_1| - |B_1 \cap S_0| - |B_1 \cap U^{\mathcal{C}_{\geq 3}^2}|) \geq$

$2(|B_1| - |S_0| - m_3)$. Hence, by (6)

$$2|B_1 - (S_0 \cup U^{\mathcal{C}_{\geq 3}^2})| \geq \begin{cases} m_{11} + 1, & \text{(a) if } m_{12} = 0, 1, m_2 = 0; \\ m_{11} + 2m_2 - 2 \geq m_{11} + m_2, & \text{(b) if } m_{12} = 0, m_2 \geq 2; \\ m_{11} + 2m_{12} - 1 \geq m_{11} + m_{12}, & \text{(c) if } m_{12} \geq 2 \text{ and } m_2 = 0; \\ m_{11} + 2m_{12} + 2m_2 - 2 \geq m_{11} + m_{12} + m_2, & \text{(d) if } m_{12} \geq 1 \text{ and } m_2 \geq 2. \end{cases}$$

On the other hand,

$$\sum_{y \in A \cap V(M) \cup \{y^*\}} |\mathcal{C}_{1y}| \leq |\mathcal{C}_1| = |\mathcal{C}_{11}^1| + |\mathcal{C}_{11}^2| + |\mathcal{C}_{12}| \leq \begin{cases} m_{11} + 1, & \text{(a) if } m_{12} = 0, 1 \text{ and } m_2 = 0; \\ m_{11} + m_2, & \text{(b) if } m_{12} = 0; \\ m_{11} + m_{12}, & \text{(c) if } m_2 = 0; \\ m_{11} + m_{12} + m_2, & \text{(d) otherwise.} \end{cases}$$

So $-|B_1 \cap S_0| - 2|B_1 - (S_0 \cup U^{\mathcal{C}_{\geq 3}^2})| + \sum_{y \in A \cap V(M) \cup \{y^*\}} |\mathcal{C}_{1y}| \leq 0$, and thus from inequalities (7) and (8), we get

$$e_H(A, B) \leq \sum_{\substack{xy \in M \\ x \in B, y \in A}} d_H(x) < \sum_{\substack{xy \in M \\ x \in B, y \in A}} d_H(y) + d_H(y^*) \leq e_H(A, B),$$

achieving a contradiction. □

The proof of Theorem 1.2 is then completed. ■

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